# Pigeonhole Principle

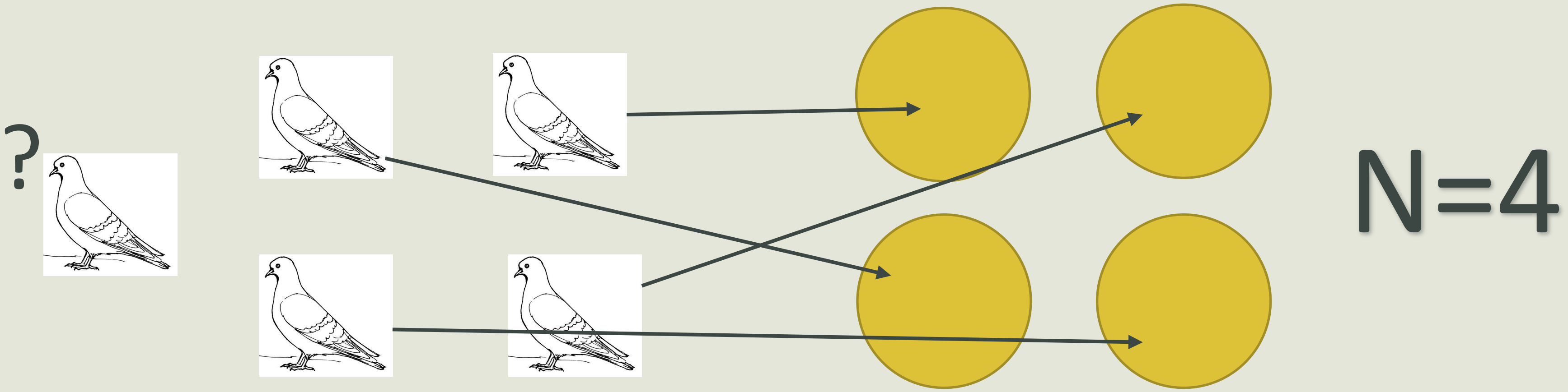
Lecture 1 Jan 10, 2021

# Pigeonhole Principle

- **Version 1**: If N+1 or more pigeons are placed in N holes, then one hole must contain 2 or more pigeons

OR: If N+1 THINGS are labeled with N labels, then at least two THINGS will have the same label

For example, if 5=4+1 (or more) pigeons sit on 4 nests, then at least 2 of them will end up on the same nest



# More general version

**Version 2**: If MN+1 or more THINGS are labeled with N labels, then at least M+1 things will have the same label

For example, if M=3 and N=3, the statement reads: If you label 3x3+1=10 objects (or more) with 3 labels, then at least M+1=4 (or more) of them have the same label.

# Some Straightforward Examples*

**Q1.** If John has 10 red, blue, yellow, and black socks in a drawer, how many socks must John pull out of the drawer to guarantee he has a pair (same color)?

**Answer:** Here the labels are the N=4 colors: RED, BLUE, YEllOW, BLACK

By Pigeonhole-Principle (PP), if John takes out 5=4+1 socks, then at least 2 of them will have the same color.

By Version 2, since 10=2x4+2>2x4+1, if he takes all the socks out, at least 3=2+1 of them will have the same color.

* Questions are from:

- https://artofproblemsolving.com/wiki/index.php/Pigeonhole_Principle
- https://math.berkeley.edu/~rhzhao/10BSpring19/Worksheets/Discussion%203%20Solutions.pdf
- http://pi.math.cornell.edu/~bux/teaching/Putnam/2003/w04.pdf

# Some Straightforward Examples *

**Q2.** Show that in a 8×8 chess board, it is impossible to place 9 rooks so that they all don't threaten each other

**Answer:** Two rooks on a chess board threaten each other if they are in the same row OR same column. So we may just consider the rows as the labels: the 8 rows are the N=8 labels.

If have N+1=9 rooks, each labeled by the corresponding row, by PP, two of them have the same label, meaning that they belong to the same row. So it is impossible to place 9 rooks so that they all don't threaten each other

# A bit twisted Examples

**Q3**. Suppose S is a set of N+1 integers. Prove that there exists distinct $a, b \in$ S such that $a - b$ is divisible by N

**Answer.** Here we need some facts from number theory.

Fact 1: The possible remainders of an integer, when dividing by N, are 0, 1, 2, …, N-1. So there are N possibilities.

If S is a set of N+1 numbers, by PP, two of the numbers in the set S, call them $a, b \in$ S, will have the same remainder $r \in \{ 0, 1, \ldots, N-1\}$ when dividing by N.

Fact 2: $a, b$ have remainder $r$ means $a = x\,N + r$ and $b = y\,N + r$ .

Therefore, $a - b = (x - y)N$ is divisible by N.

# A bit twisted Examples

**Q4**:  Show that in any group of N people, there are two who have the same number of friends within the group

**Answer.**  The number of friends for each person is a number between 0 and N-1 (That is N possibilities). So we have N people labeled with N possibilities. We are not in the setting of PP yet. There are two possible scenarios:

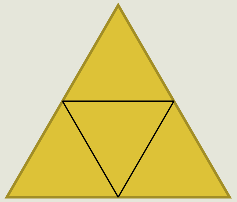
1) All the N possibilities are taken, meaning that for each number $0 \leq a \ \leq N-1$, there is a person with $a$ friends. In particular, there is a person with 0 friend and another person with N-1 friends. That is impossible! (Why?)

2) Less than all the N possibilities are taken. So at most N-1 of N possibilities are taken. Then by PP, since N= (N-1)+1, there are two people with the same number of friends.

# A bit twisted Examples

**Q5.** Given five points inside an equilateral triangle of side length 2, show that there are two points whose distance from each other is at most 1.

**Answer.** Divide such an equilateral triangle of side length 2 to 4 smaller equilateral triangles of side length 1 as in the following picture. By PP, two of those 5 points live inside one of these smaller triangles. It is left to show that every two points inside an equilateral triangle of side length 1 have distance at most 1 (Try to prove this rigorously).



# A bit more twisted Example

**Q6.** A soccer team scores 30 goals in 20 games, scoring at least one goal in each game. Show that there is a number of consecutive games where the team has scored exactly 9 goals.

**Hint:** For $i = 1, \dots, 20$ , let $s_i$ denote the number of goals scored from game 1 up to game $i$.

For example, $s_{20} = 30$.

The number of goals scored since game $i$ until game $j$ is the difference $s_j - s_i$ .

So the question is asking you to show that there are $1 \leq i < j \leq 20$ such that $s_j - s_i = 9$

Think about this and I will finish it next time.

# COMMENT

Try to spend more time on sample questions from those links or anywhere else to get more comfortable with this subject

Often, We Will Need To Know More Than Just The Topic We Are Discussing

**Don’t Worry,** I Will Explain Everything Needed

If You Encountered Such a Problem, Google Search, Don’t Give Up!